Numerical computation of the scattering matrix of an electromagnetic resonator

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A method is presented to investigate diffraction of an electromagnetic plane wave by an infinitely thin infinitely conducting circular cylinder with longitudinal slots. It is based on the use of the combined boundary conditions method that consists of expressing the continuity of the tangential components of both the electric and the magnetic fields in a single equation. This method proves to be very efficient for this kind of problem and leads to fast numerical codes. The scattering matrix that is obtained from this theory can then be used in a multiscattering method to study wave propagation in square arrays of such resonators with an emphasis on the low-frequency behavior.

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I. INTRODUCTION

It has recently been shown that periodic arrays of metallic rods, i.e., two-dimensional metallic photonic crystals [1–4], could exhibit extremely unusual properties with respect to wave propagation [5–7]. Some of these artificial materials have been called "left-handed" because they behave as if they had negative permeability and permittivity [8–12]. The geometry of the scatterers at issue is that of resonator, made of layered curved strips. In the present work we present a numerical method to compute the scattering matrix of such a curved striped device. This scattering matrix can then be used to study numerically wave propagation inside a finite array of such resonators. This numerical study will be presented in a future work [13], we just give here a brief exposition of the numerical method.

The problem of the penetration of electromagnetic waves in a conducting circular cavity through a narrow axial aperture has been treated by several authors. Several methods have been used to achieve the determination of the field inside the cavity. Beren [14] used the aperture field integral equation, the electric field integral equation, and H-field integral equation to determine the field around an axially slotted cylinder, while Johnson and Ziolkowski [15] gave a generalized dual series solution for this problem. Mautz and Harrington treated the field penetration inside a conducting circular cylinder through a narrow slot in both TE [16] and TM [17] polarizations. More recently Shumpert and Butler [18,19] proposed three methods to study the penetration in conducting cylinders. In this article, we propose a method to calculate the field inside and around a slotted circular cavity with longitudinal slots. It is based on the combined boundary conditions method introduced first by Montiel and Nevière [20,21]. Section II is dedicated to the description of the theory. In Sec. III we give some details about the numerical scheme and then compare our results with previous work. We also give some preliminary results on the electromagnetic behavior of an array of resonators.

II. SCATTERING THEORY

The structure under study is depicted in Fig. 1. The space is divided into two regions: region 1 (exterior region: $r>R$) and region 2 (interior: $r<R$) that are assumed to be dielectric and homogeneous with relative dielectric permittivities ε_1 and ε_2, respectively. On the interface between these two media are deposited a finite number of infinitely conducting, infinitely thin circular strips that are invariant along the z direction. The device is illuminated by a TMz (electric field parallel to the z axis) or TEz (magnetic field parallel to the z axis) monochromatic electromagnetic wave under incidence θ_0 with vacuum wavelength λ.

Throughout this paper we assume an $\exp(-i\omega t)$ time dependence. The z component of the electric or the magnetic field will be denoted by $u(\theta,r)$. We denote by Ω_1 the union of the strips and by Ω_2 its complementary in $[0,2\pi]$.

In the exterior region we express the total field as

$$u_1(\theta,r)=\sum_{n\in\mathbb{Z}} a_n J_n(k_1 r)\exp(in\theta) + \sum_{n\in\mathbb{Z}} b_n H_n^{(1)}(k_1 r)\exp(in\theta). \quad (1)$$

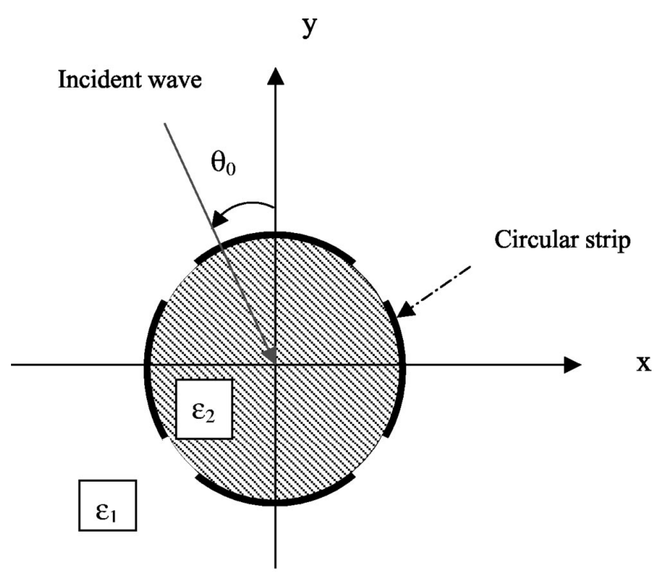

FIG. 1. Geometry of the problem: a TEz or a TMz polarized plane wave illuminates the slotted cylinder.

Likewise in the interior region the total field may be expressed as

$$u_2(\theta,r)=\sum_{n\in\mathbb{Z}} c_n J_n(k_2 r)\exp(in\theta), \tag{2}$$

where a_n, b_n, and c_n are the amplitudes of the incident, the diffracted and the transmitted waves, respectively. We denote J_n and $H_n^{(1)}$ the Bessel and the Hankel functions of the first kind. $k_p=k_0\sqrt{\varepsilon_p}=(2\pi/\lambda)\sqrt{\varepsilon_p}$, with $p=1,2$ and $\mathbb{Z}$ denotes the usual set of relative integers.

Amplitudes a_n being known, the problem is to determine amplitudes b_n and c_n from which the total field can be calculated everywhere. For that purpose one must write the boundary conditions at the interface between both dielectric media. This is done in the next subsections by distinguishing the TMz and the TEz cases of polarization.

A. TMz polarization

The boundary conditions applied to the tangential components of the electromagnetic field at the interface defined by $r=R$ lead to

$$u_1(\theta,R)=u_2(\theta,R),\quad \forall\theta\in[0,2\pi], \tag{3a}$$

$$\left(\frac{du_1}{dr}\right)_{(\theta,R)}=\left(\frac{du_2}{dr}\right)_{(\theta,R)},\quad \forall\theta\in\Omega_2. \tag{3b}$$

With the supplementary condition that the electric field must vanish on the strips

$$u_1(\theta,R)=u_2(\theta,R)=0,\quad \forall\theta\in\Omega_1. \tag{4}$$

Following Montiel and Nevière [20,21], Eqs. (3b) and (4) can be combined in a single equation that holds for every θ in $[0,2\pi]$:

$$[1-\chi(\theta)]u_2(\theta,R)+g\chi(\theta)\left[\left(\frac{du_2}{dr}\right)_{(\theta,R)}-\left(\frac{du_1}{dr}\right)_{(\theta,R)}\right]=0,\ \forall\theta\in[0,2\pi], \tag{5}$$

where $\chi(\theta)$ is the characteristic function of set Ω_2,

$$\chi(\theta)=\begin{cases}1 & \text{if } x\in\Omega_2,\\ 0 & \text{elsewhere,}\end{cases}$$

and g is some numerical parameter introduced for dimensional and numerical purposes. Remark that the set of Eqs. (3a), (3b), and (4) is equivalent to the set of Eqs. (3a) and (5). Since $\chi(\theta)$ is 2π periodic it can be expanded in Fourier series

$$\chi(\theta)=\sum_{p\in\mathbb{Z}}\chi_p\exp(ip\theta). \tag{6}$$

Reporting Eqs. (1) and (2) into Eq. (3a) and projecting on the $[\exp(in\theta)]_{n\in\mathbb{Z}}$ basis gives

$$a_nJ_n(k_1R)+b_nH_n^{(1)}(k_1R)=c_nJ_n(k_2R),\quad \forall n\in\mathbb{Z}. \tag{7}$$

Then reporting Eqs. (1), (2), and (6) into Eq. (5) and projecting on the $[\exp(in\theta)]_{n\in\mathbb{Z}}$ basis leads to

$$c_nJ_n(k_2R)-\sum_{p\in\mathbb{Z}}\chi_{n-p}c_pJ_p(k_2R)+g\sum_{p\in\mathbb{Z}}\chi_{n-p}\{k_2[c_pJ_p'(k_2R)]-k_1[a_pJ_p'(k_1R)+b_pH_p^{(1)\prime}(k_1R)]\}=0,\quad\forall n\in\mathbb{Z}, \tag{8}$$

where the primes denote derivation with respect to r. From Eq. (7) one can extract c_n,

$$c_n=\frac{J_n(k_1R)}{J_n(k_2R)}a_n+\frac{H_n^{(1)}(k_1R)}{J_n(k_2R)}b_n,\quad\forall n\in\mathbb{Z}, \tag{9}$$

and report its expression into Eq. (8) to obtain the following linear system linking the amplitudes a_n and b_n:

$$\begin{aligned}a_nJ_n(k_1R)&+\sum_{p\in\mathbb{Z}}\chi_{n-p}a_p\times\left[-J_p(k_1R)+gk_2\frac{J_p(k_1R)}{J_p(k_2R)}J_p'(k_2R)-gk_1J_p'(k_1R)\right]\\&=-b_nH_n^{(1)}(k_1R)+\sum_{p\in\mathbb{Z}}\chi_{n-p}b_p\left[H_p^{(1)}(k_1R)-gk_2\frac{H_p^{(1)}(k_1R)}{J_p(k_2R)}J_p'(k_2R)+gk_1H_p^{(1)\prime}(k_1R)\right].\end{aligned} \tag{10}$$

The solution of the linear system (10) gives the unknown amplitudes b_n and then Eq. (9) gives the amplitudes c_n. Thus the field can be computed everywhere in space using Eqs. (1) and (2).

B. TEz polarization

For this case of polarization, the continuity of the tangential components of the electromagnetic field at the interface defined by $r=R$ leads to

$$\frac{1}{\varepsilon_1}\left(\frac{du_1}{dr}\right)_{(\theta,R)}=\frac{1}{\varepsilon_2}\left(\frac{du_2}{dr}\right)_{(\theta,R)},\quad\forall\theta\in[0,2\pi], \tag{11a}$$

$$u_1(\theta,R)=u_2(\theta,R),\ \forall\theta\in\Omega_2. \tag{11b}$$

With the supplementary condition that the electric field must vanish on the strip

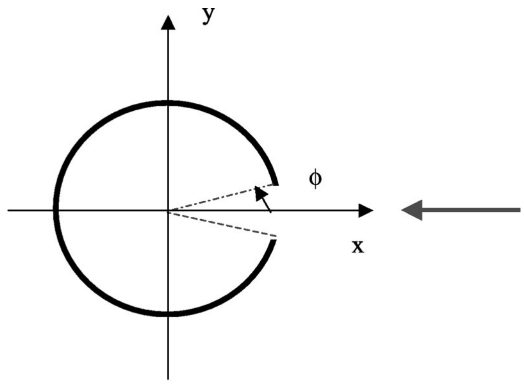

FIG. 2. Electromagnetic penetration into a circular cavity through a narrow slot.

$$\frac{1}{\varepsilon_1}\left(\frac{du_1}{dr}\right)_{(\theta,R)}=\frac{1}{\varepsilon_2}\left(\frac{du_2}{dr}\right)_{(\theta,R)}=0, \quad \forall\,\theta\in\Omega_1. \qquad (12)$$

Here again we can replace Eqs. (11b) and (12) by

$$[1-\chi(\theta)]\frac{1}{\varepsilon_2}\left(\frac{du_2}{dr}\right)_{(\theta,R)}+g\chi(\theta)[u_2(\theta,R)-u_1(\theta,R)] =0, \ \forall\,\theta\in[0,2\pi]. \qquad (13)$$

Reporting Eqs. (1) and (2) into Eq. (11a) and projecting on the $[\exp(in\theta)]_{n\in\mathbb{Z}}$ basis gives

$$a_nJ_n'(k_1R)+b_nH_n^{(1)\prime}(k_1R)=\frac{k_2}{k_1}\frac{\varepsilon_1}{\varepsilon_2}c_nJ_n'(k_2R), \quad \forall n\in\mathbb{Z}. \qquad (14)$$

We remark that the set of Eqs. (11a), (11b), and (12) are equivalent to the set of Eqs. (11b) and (13). Reporting Eqs. (1), (2), and (6) into Eq. (13) and projecting on the $[\exp(in\theta)]_{n\in\mathbb{Z}}$ basis leads to

$$\frac{k_2}{\varepsilon_2}c_nJ_n'(k_2R)-\frac{k_2}{\varepsilon_2}\sum_{p\in\mathbb{Z}}\chi_{n-p}c_pJ_p'(k_2R) +g\sum_{p\in\mathbb{Z}}\chi_{n-p}\{c_pJ_p(k_2R) -[a_pJ_p(k_1R)+b_pH_p^{(1)}(k_1R)]\}=0, \quad \forall n\in\mathbb{Z}. \qquad (15)$$

From Eq. (14) one can extract c_n,

$$c_n=\frac{k_1}{k_2}\frac{\varepsilon_2}{\varepsilon_1}\left(\frac{J_n'(k_1R)}{J_n'(k_2R)}a_n+\frac{H_n^{(1)\prime}(k_1R)}{J_n'(k_2R)}b_n\right), \quad \forall n\in\mathbb{Z}, \qquad (16)$$

and report its expression into Eq. (15) to obtain the following linear system linking the amplitudes a_n and b_n:

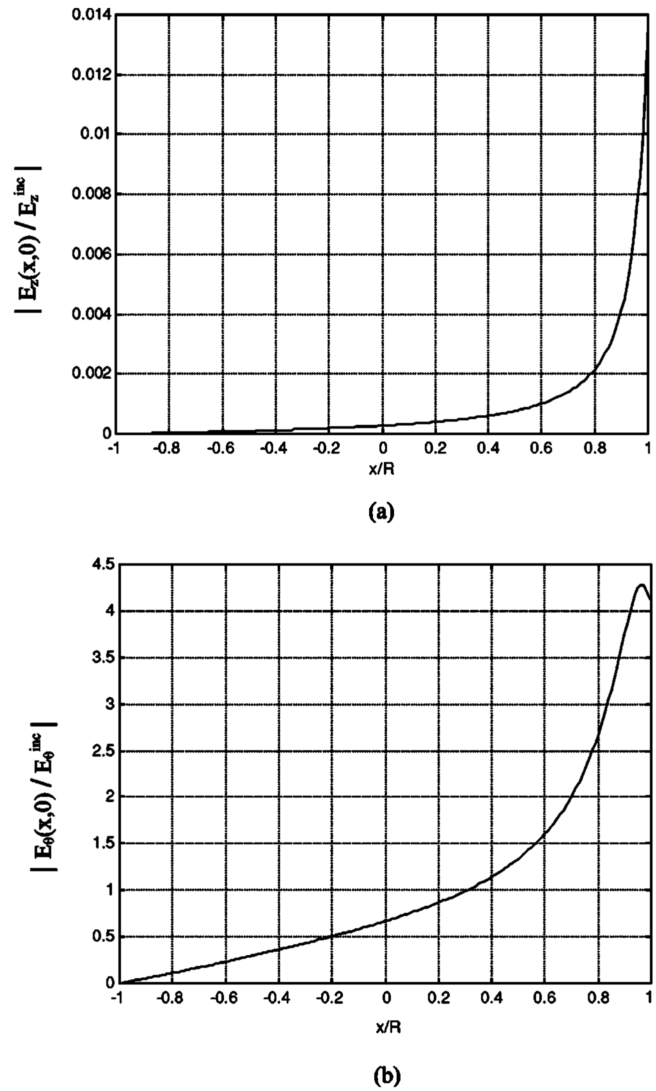

FIG. 3. (a) Magnitude of normalized electric field on the x axis of a slotted circular cylinder excited by TM_z plane wave $\{k_1R=0.7,\theta_0=180°,\phi=5°\}$. (b) Magnitude of normalized electric field on the x axis of slotted circular cylinder excited by TE_z plane wave $\{k_1R=0.7,\theta_0=0°,\phi=5°\}$.

$$a_n\frac{k_1}{\varepsilon_1}J_n'(k_1R)+\sum_{p\in\mathbb{Z}}\chi_{n-p}a_p\left[-\frac{k_1}{\varepsilon_1}J_p'(k_1R) +g\frac{k_1}{k_2}\frac{\varepsilon_2}{\varepsilon_1}\frac{J_p(k_2R)}{J_p'(k_2R)}J_p'(k_1R)-gJ_p(k_1R)\right] =-b_n\frac{k_1}{\varepsilon_1}H_n^{(1)\prime}(k_1R)+\sum_{p\in\mathbb{Z}}\chi_{n-p}b_p\left[\frac{k_1}{\varepsilon_1}H_p^{(1)\prime}(k_1R) -g\frac{k_1}{k_2}\frac{\varepsilon_2}{\varepsilon_1}\frac{J_p(k_2R)}{J_p'(k_2R)}H_p^{(1)\prime}(k_1R)+gH_p^{(1)}(k_1R)\right]. \qquad (17)$$

The solution of the linear system (17) gives the unknown amplitudes b_n and then Eq. (16) gives the amplitudes c_n. Thus the field can be computed everywhere in space using Eqs. (1) and (2).

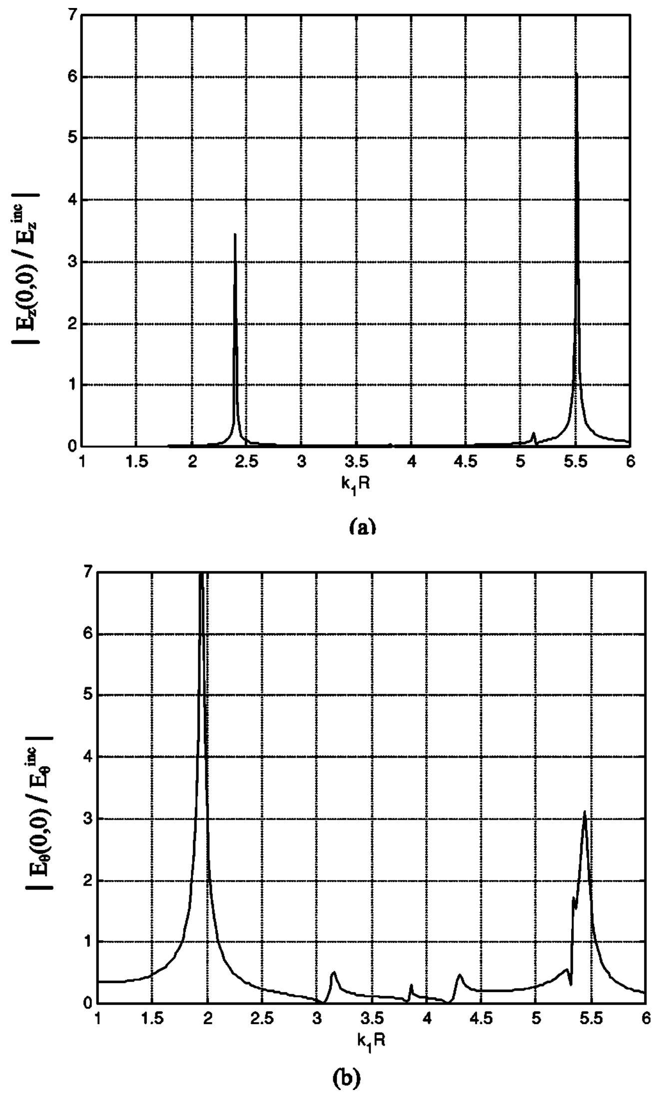

FIG. 4. (a) Normalized electric field amplitude at the center of the cylinder for various k_1R with $(\theta_0=0°, \phi=5°)$. (b) Normalized electric field amplitude at the center of the cylinder for various k_1R with $(\theta_0=0°, \phi=5°)$.

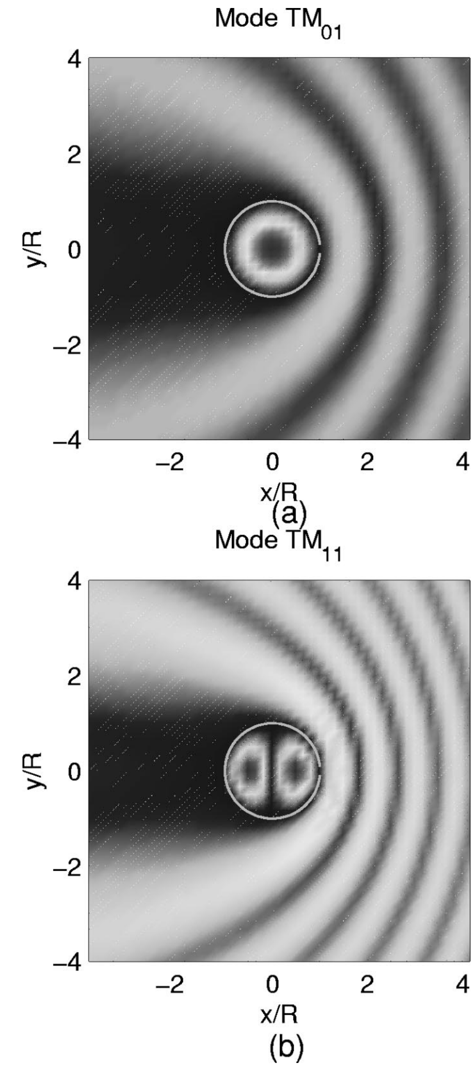

FIG. 5. Map of the electric field for values of k_1R corresponding to the modes: (a) TM_{01} and (b) TM_{11} of the cylinder.

III. NUMERICAL RESULTS

The infinite linear systems (10) and (17) are truncated to a finite size by retaining only $(2N+1)$ coefficients and solved to obtain a representation of the field at truncation order N. The convergence of the results has been checked by increasing integer N and using the usual criteria of energy balance (optical theorem) and reciprocity. We have also verified that the boundary conditions are fulfilled, for instance the nullity of the tangential electric field on the strips. In all the calculations carried in this paper we set $g=-10^{-3}$. However, as mentioned in Ref. [22], numerical experiments show that only the sign of g is of importance: the numerical scheme is more stable with a negative value of g. All the computations reported have been obtained on a Personal Computer (200 MHz processor with 32 Mo of RAM), only a few seconds are necessary to perform each result shown here.

In the following we provide some numerical examples and compare our results with those obtained in previous works [16–19]. In our first example, we consider a circular cavity with a single longitudinal narrow slot (see Fig. 2) with $\phi=5°$ and we compute the interior field on x axis. Figures 3(a) and 3(b) show the magnitude of the normalized electric field in both the TM_z and the TE_z cases of polarization. It can be seen that our results are in excellent agreement with those published recently by Shumpert and Butler [18,19] see, for instance, Fig. 6 in the last reference. In the second example, we consider a circular cavity with an aperture such that $\phi=5°$. In Figs. 4(a) and 4(b) are plotted the normalized electric field amplitude at the center of the cylinder for various values of the parameter k_1R for both the TM_z and the TE_z cases of polarization. These curves agree with those obtained by Mautz and Harrington [16,17]. It is worth noticing that the resonances in these plots correspond to the modes of the cavity.

Finally we give the map of the electric field around and inside the slotted cylinder when excited by a plane wave such that k_1R corresponds to a mode of the closed cylinder. We can see in Figs. 5(a) and 5(b) that the modes

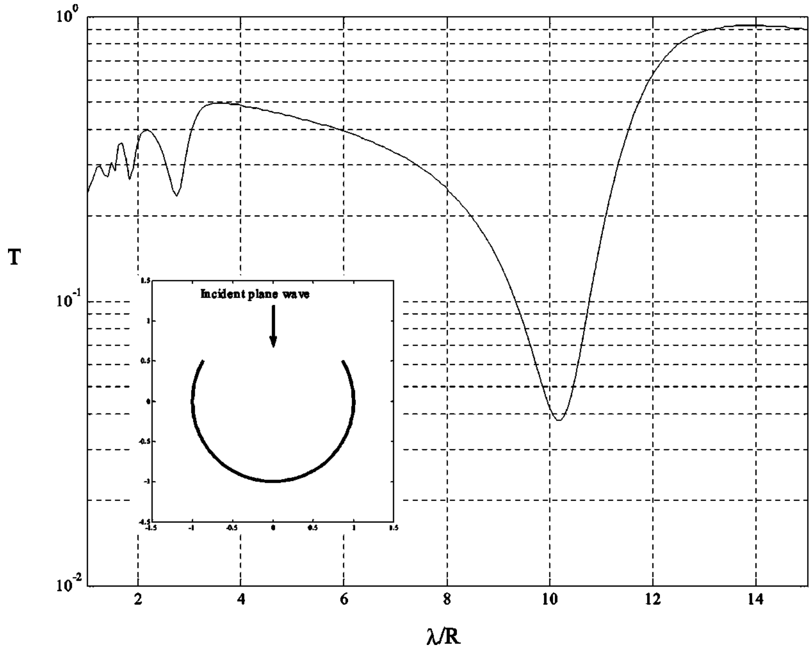

FIG. 6. Transmission through a resonator with an opening of 120°. The inset represents the resonator. The peaks correspond to the natural resonances.

$TM_{01}(k_1R=2.404)$ and $TM_{11}(k_1R=3.832)$ are excited inside the structure.

We give now some preliminary results concerning wave propagation inside a periodic array of the resonators described above. We consider an array of 7×7 resonators with square symmetry. The opening angle of the strip is 120°. We have plotted in Fig. 6 the transmission of one resonator, the transmission being defined as the flux of the Poynting vector on a segment situated below the photonic crystals. The deep peaks correspond to the resonances where the field is strongly localized in the cavity. In order to compute the scattered field, we use a multidiffraction technique described in Refs. [23,24].

We plot in Fig. 7 the transmission versus the wavelength through the device, when it is illuminated by a plane wave. As a comparison, we have also plotted the transmission obtained for perfectly conducting rods with the same radius as that of the resonators. We clearly see the opening of gaps near the resonances, a phenomena that had already been suggested [25,26].

IV. CONCLUSION

We have developed a very efficient and fast method adapted to study diffraction of an electromagnetic wave by a finite number of infinitely thin, infinitely conducting strips deposited on a dielectric cylinder. It is based on the combined boundary conditions method. The method is very

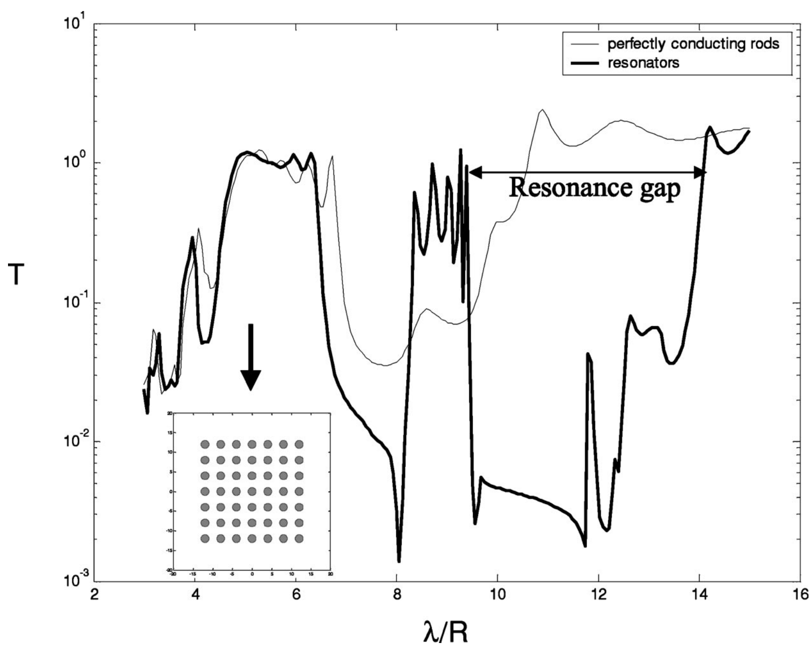

FIG. 7. Transmission through a square array of resonator (solid line), and the same array with perfectly conducting rods (light line). Both resonators and rods have the same radius. The inset shows the structure.

simple to implement. The numerical examples that have been given to illustrate the method are not restrictive. One can use as an incident radiation a beam of any shape. It suffices to calculate its corresponding incident amplitudes a_n. It is also possible to study the radiation pattern of a source located at the center of the cylinder by making slight changes in the equations. In the near future, we intend to use this scattering theory to study wave propagation in an array of resonators.

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